COUNTEREXAMPLES IN TOPOLOGY GENERATED BY LARGE CARDINALS, PART I

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Abstract. We generate some counterexamples in General Topology using Large Cardinals, particularly Large Cardinals with elementary embedding and filter characterization, and models of Set Theory containing such Large Cardinals (particularly V). Embeddings and Filers/Ultrafilters from Large Cardinals (and the crit. points of their respective embeddings) are used to create topological counterexamples not possible in "regular" ZFC or ZF; in particular, we show that the Real Line and its Stone-Cech Compactification is preserved under Large Cardinals not inconsistent with AC, and that the Strong Ultrafilter Topology is not preserved under Vopenka's principle, and two other more such Topological examples. Such counterexamples are given by Stenn and Seeback in their "Counterexamples in General Topology". This paper also generates the adequate machinery to deal with Large Cardinals and their interaction with non-logical fields, particularly with embeddings and their such interactions. More counterexamples will be worked on in another paper.

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Date: May 2025.
2020 *Mathematics Subject Classification.* 03E55, 54G20, 54C25, 54D99.

1. Introduction

We start off with one example in particular: homogeneity of spaces. A space X is *homogeneous* if for any pair of points of the space there is an autohomeomorphism of the space mapping one point to the other. Now, such a definition can quite easily be constructed in ZFC, and some examples can be given, such as the Hilbert cube $[0,1]^\omega$, the compactification of the real line $\beta\mathbb{R}$, and if X is a first-countable zero-dimensional space then X^ω is homogeneous (Dow and Pearl, extension of Lawrence's Theorem). But an interesting dynamic emerges: It is known that filters play an role in homogeneity, with one role which is if $\mathcal{B}$ is an ultrafilter, then for a pair $(a,b) \in \mathcal{B}$, assuming well-order, then a well-order of points and pairs in $\mathcal{B}$ can be undertaken, and thus if $b \subseteq \mathcal{B}$, given b is solely comprised of pairs $(a,b) \in \mathcal{B}$ and is also a filter, b is homogeneous. b can also be of any cardinality. (*Theorem 1.*)

This holds pretty elegantly and neatly in a "usual" notion of ZF, but given a measurable cardinal, especially with its "ultrapower formulation", which is:

(1) There exists a κ-complete nonprincipal ultrafilter on κ.
(2) There exists a nontrivial elementary embedding $j : V \to M$ with M transitive and κ is the critical point of the embedding (least ordinal moved by the embedding).
(3) There exists a nonprincipal ultrafilter U such that the ultrapower $(Ult_U(V), \in_U)$ of the universe is well-founded.

in which the above are all equivalent. "κ-completeness" of a filter is whenever $\gamma < \kappa$, and $\{A_\alpha : \alpha < \gamma\} \subseteq \mathcal{B}$, in which $\mathcal{B}$ is a filter, then $\bigcap_{\alpha<\gamma} A_\alpha \in \mathcal{B}$. In other words, it is a "limitation" of the elements of $\mathcal{B}$ to κ. We proceed via Ultrapowers, which are denoted by $Ult_U(V) = \{[f] : f \in \Pi_{i\in X} M_i\}$, and $\in_U$ if the elements "obeying" the relation are in U, with all M_i (models) being equal. A consequence of measurable ultrapowers on homogeneous subsets of $\mathcal{B}$ is that there is very little individuality of homogeneous subsets of $\mathcal{B}$; most (if not all) are of the same cardinality, because elements of the ultrapower are more-or-less U-similarized. Therefore, we have proven *Theorem 1.* This is a prime example of Large Cardinals generating a topological counterexample.

A more down-to-earth example is that of M-ultrafilters and the non-convergence of (to a point) sequences of the Meagre Sets of the Real Line. Essentially, an M-ultrafilter is a subset of the ultrafilter such that subsets of the power set of κ are in M.

2. Definitions, A brief overview of Large Cardinals

Definition 2.1 (Large Cardinals). A **Large Cardinal Axiom** is an axiom added to Set Theory (generally ZFC) such that it implies the existence of a cardinal κ which is "very large"; generally larger than the least α such that $\omega_\alpha = \alpha$. It also cannot be proved or refused using ZF(C); it is independent of ZF(C).

With the advent of Large Cardinals, comes the introduction of the Large Cardinal Hierarchy, a Hierarchy of Large Cardinal Axioms based on axiom strength, particularly consistency strength. Essentially, a Large Cardinal Axiom is stronger than another Large Cardinal Axiom if the former axiom can prove the latter, or

ZF or ZFC with the former Large Cardinal Axiom proves that the latter Large Cardinal Axiom + ZF(C) is consistent.

Definition 2.2 (Elementary Embeddings). An **Elementary Embedding** between two structures (which are sets with finite operations and relations), or models M and N of the same signature σ is a map $h : N \to M$ such that for every first-order formula $\phi(x_1, .., x_n)$, in which $x_1, ..., x_n \in N$, $N \models \phi(x_1, ..., x_n)$ if and only if $M \models \phi(h(x_1), ..., h(x_n))$.

A *signature* of a model or structure is the set of operations and relations of said model or structure. In this paper, we focus mainly on Large Cardinals given by elementary embeddings. An example of a Large Cardinal given by elementary embeddings is a measurable cardinal in (2). Another example is of *weakly compact cardinals*, cardinals (**Definition 2.3**) which are again may be given by elementary embeddings[1]: A cardinal κ is said to be *weakly compact* iff for every $A \subset \kappa$, there is a transitive set M with cardinality κ and $\kappa \in M$ such that there is a transitive set N with an elementary embedding $j : M \to N$ with critical point κ.

A stronger axiom is (obviously) *strongly compact cardinals*[2], a cardinal κ is θ-strongly compact iff there is an elementary embedding $j : V \to M$, V is the set-theoretic universe constructed from the power set relation, and M is transitive class. j has critical point κ, such that $j'\theta \subset s \in M$ for some set $s \in M$ with $|s|^M < j(\kappa)$. Although they have a characterization via embeddings, trees are probably the most elegant and straightforward method to use in Topology; after all, trees are not uncommon in Topology. As an example of a strongly compact cardinal satisfaction within Set Theory, they satisfy Tychonoff's theorem. Other potential Large Cardinals to be used are Reinhardt, Rank-into-Rank, and super-compact cardinals, mainly to be used to study embeddings and their interaction with counterexamples in Topology.

2.1. **A proof of Tychonoff's Theorem from Strongly Compact Cardinals.**

Theorem 2.1. *Strongly compact cardinals satisfy Tychonoff's Theorem in ZFC.*

Proof. Let Tychonoff's Theorem be equivalent to the statement that the product of $\aleph_\alpha$-compact spaces is $\aleph_\alpha$-compact under the product topology (In the original "theorem sketch" proposed by "Cantor's Attic", this is $\aleph_0$, but this can easily be extended to all ordinals $\alpha < \kappa$). The proof essentially boils down to that (1) the product of κ-compact spaces should itself be κ-compact, and κ is strongly compact and (2) the product of κ-compact spaces is strongly compact. We proceed via

[1]Weakly compact cardinals are very diverse in their modes of characterization, but said modes include:

(1) Satisfaction of the weak compactness theorem for $L_{\kappa\kappa}$, in which $L_{\kappa\kappa}$ is an infinitary language, and κ is inaccessible.
(2) Let M contain at most κ-many subsets of κ, and this implies a κ-complete nonprincipal filter measuring every set in M. κ is then weakly compact.

Characterization (1) might be used more in this paper, particularly for its relation to compactness, even in General Topology. Note that there are more characterizations of weakly compact cardinals, but not much attention is paid to them.

[2]Note that they are unusually strong, and are much stronger than weakly compact cardinals.

embeddings. Transitivity has no effect on the theorem. Let the spaces in (1) be in V. Under the embedding, the cardinality of the sub-covers of the open covers is preserved. As κ is a critical point of the embedding, cardinality of subcovers of open covers $< \kappa$ is preserved. To products - Let the products of κ-compact spaces under the embedding not be κ-compact. But only $< \kappa$-compactness is preserved under the embedding. (2) Let h and i make up the injections comprising the embedding j. Then, we can have h and i manually "lift" the product space and topology to the other model. In the product of κ-compact spaces, take the open covers. A large portion of the proof of (1) is used, but is applied to individual injections. For extension into finite-ness[3], note that strongly compact cardinals, particularly embeddings, do not collapse finite sets and subcovers (Note that the original restriction to $\aleph_0$ makes this collapse less tedious, but the collapse can still be done with $\aleph_\alpha$). □

Remark 1. When individual injections of embeddings start to be defined, there are two options: (1) use a second-order theory, or (2) make the embedding a schema. (2) will be used much more often, to avoid the hassle of topological definitions in second-order logic. Originally, the idea of using Vopenka cardinals as a key cardinal in this paper was proposed, but was quickly taken down to the second-order nature of it.

2.2. **Other Large Cardinals.**

Definition 2.3. A *Reinhardt Cardinal* is the critical point of an elementary embedding $j : V \to V$ of the set-theoretic universe V to itself.

Note that Reinhardt cardinals are inconsistent with ZF (per Kunen's inconsistency).

Definition 2.4. A *Rank-into-Rank Cardinal* is the critical point of an elementary embedding $j : V_\lambda \to V_\lambda$ for an ordinal λ.

Remark 2. A simple method for formulating large cardinal axioms is in terms of elementary embeddings. Basically, we have an embedding from usually a, not the set-theoretic universe to another, generally transitive model to another model. Such a large cardinal is the critical point of such an embedding (courtesy of the SEP).

Definition 2.5. *Supercompact cardinals* are cardinals which, given λ a cardinal, κ is λ-supercompact if there exists an elementary embedding $j : V \to M$, M transitive, with κ a crit. point of j, and $M^\theta \subset M$, M is closed under arbitrary sequences of length θ. κ is *supercompact* if it is θ-supercompact for all θ.[4] κ is a (slightly stronger) measurable.

Definition 2.6. (Vopenka's Principle) For any proper class of structures for the same language, there is one that is elementarily embeddible to the other.

We define a Vopenka cardinal in terms of an inaccessible cardinal κ such that $V_\kappa \models$ Vopenka's Principle.

Definition 2.7. X is Vopenka in κ iff for any natural sequence $\langle M_\alpha | \alpha < \kappa \rangle$ there is a $j : M_a \prec M_b$ for some $\alpha < \beta < \kappa$ with critical point in X.

[3] A topological space (X, τ) is said to be strongly compact if every preopen cover of (X, τ) admits a finite subcover.

[4] Generally this is applied for all $\theta > \kappa$.

and

Definition 2.8. A cardinal κ is Vopenka iff κ is Vopenka in κ.

where,

Definition 2.9. A sequence of structures $\langle M_\alpha | \alpha < \kappa \rangle$ is *natural* iff each M_α is of the form $\langle V_{f(\alpha)}, \in, \{\alpha\}, R_\alpha \rangle$ where $R_\alpha \subseteq V_{f(\alpha)}$ and $\alpha < \beta < \kappa$ implies that $\alpha < f(\alpha) \leq f(\beta) < \kappa$.

Definition 2.10. κ is said to be *η-extendible* iff there exists a ζ and a $j : V_{\kappa+\eta} \prec V_\zeta$ with $\text{crit}(j) = \kappa$ and $\eta < j(\kappa)$. κ is *extendible* iff κ is η-extendible for every $\eta > 0$.

Definition 2.11. A cardinal κ is said to be *η-extendible* iff there is an elementary embedding $j : V_{\kappa+\eta} \to V_\theta$, with crit. point κ, for an ordinal θ.

Definition 2.12. A *tree* is a poset $(T, <)$ such that, for each $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by the relation $<$.

Trees are particularly relevant to the discussion of Large Cardinals in Set Theory because of the fact that working with posets (particularly given a topological sense, and even a model-theoretic sense) are made easier and more elegant.

Definition 2.13. A *Suslin tree* typically refers to a ω_1-Suslin tree. For any regular κ, a κ-Suslin tree is a κ-tree in which all chains and all antichains have cardinality below κ.

Definition 2.14. An *Aronszajn tree is a tree of height $\aleph_1$ with no $\aleph_1$-branches and no $\aleph_1$-levels.*

Definition 2.15. A cardinal κ has the *tree property* if there are no κ-Aronszajn trees.

Definition 2.16. To define a "tree topology", one must note that when the tree is finite, the tree's topology coincides with the product topology on subsets of the posets. Thus a topology on trees is inspired from the Euclidean topology; open-ness of subsets of T is inspired from open-ness of subsets of $\mathbb{R}$.[5]

3. Trees and Large Cardinals

3.1. **Embeddings and Trees.** We begin by defining a notion of elementary embeddings (of Large Cardinals) in terms of trees, more specifically embeddings from trees to another set. As an example, start with measurable cardinals and the real line. Let T_κ, the said (κ-Aronszajn, κ analog of $\aleph_1$-Aronszajns) tree generated from a Large Cardinal.[6] The Real Line in this case is constructed via the transitive model M. Essentially, one proceeds with the construction of the reals in the usual way; via naturals, and then to equivalence classes of naturals (rational numbers), and then via Dedekind cuts or Cauchy sequences to Real Numbers. However, this construction might be affected by M in certain ways (not due to M's transitivity, as V is also transitive):

[5]Devlin and Shelah's definition of a tree topology in their paper "Souslin Properties and Tree Topologies" is used. [4]

[6]Measurable cardinals are used as an example, given their established-ness in Topology and Set Theory, and the fact that they are the most basic and simple "large large cardinal".

(1) Is the construction of the Reals from V preserved via the elementary embedding j into M?
(2) Given that M is constructed recursively, and j is nontrivial, would κ be strictly measurable or would it be another cardinal?

Thus the construction of the Real Numbers giving an elementary embedding that produces a Large Cardinal must be very strong; even stronger than measurability. With Measurables, we can still produce and construct the Real Line using its added measures. One can construct, recursively, a hierarchy in which the naturals are represented and therefore constructed, from the power set of κ, κ measurable (competitor to V?), that satisfies κ-additivity. How could this work? Represent a hierarchy for constructing the naturals via $P(\kappa)$ by N_κ, κ represents a measurable cardinal. Then $M := M_\kappa$, in which it is "built-up" recursively via a modified version of the power set operation, but suited to fit the needs of measurable cardinals.

Remark 3. For a tree-theoretic notion of the embedding, represent V and V_κ as a tree of height κ, and the root of the tree simply as $V_0 = \emptyset$. To "represent" V in terms of posets, order V in terms of the rank of the levels via $\leq$, and well-order of the levels of V is given via $<$.

V, when represented as a tree, is a κ-tree. The same thing is done for the image of the embedding M, which is constructed via recursion.

Definition 3.1. If M is not constructed via recursion, define an arbitrary p-order (partial order) on M by assignment of arbitrary ranks (we use the Axiom of Choice inside of M) of sets (or classes) inside M.

Remark 4. The ranking bijection is all contained within M, and is preserved from V to M by k; if we proceed to define and quantify over individual injections in k, treat k as a function schema, because k (and its constituent functions) is Σ_1. Let h_n be an injection in k. Then, have h such that if r_{V_n} is a rank (n) from V_n, then it corresponds (injectively) into a rank r_{M_n}, in which it is a rank (n) from M_n. $\mathcal{L}_\alpha(r_{V_n}), \alpha = n$, and the same goes for M. *In other words, we have the preservation of trees, and ranks in trees, by schemata of injections.* Define h_n recursively in this manner, in that $h_{n_1} : r_{V_{n+1}} \to r_{M_{n+1}}$, to construct a schema of injections k. This schema is a representation to the "main embedding" j.

Ranks of sets in the above models comprise an "auxillary set" in the same model; this is a mapping from rank auxillary set of V to rank auxillary set of M. Rank-specific symbols and sets are also added to both the R, F, and C symbols (relation, function, and constant symbols) of V and M. The Axiom of Choice is obviously vital to this construction. However, given large cardinals "too large" to permit AC, a natural question emerges: *how, and in which way, would, say, Reinhardt cardinals be inconsistent with AC and therefore the ranking?* Measurable cardinals are very privileged in this case; they only require that κ is the first ordinal moved by j, and don't led to any annoying inconsistency results. An incompatibility of defining, or even preserving ranks itself with an extremely strong large cardinal must be obtained.

Definition 3.2. We define *critical points* of injection schemata as the smallest ordinal out of all the injections in the injection "class", or the whole injection class, which is not mapped to itself out of all such injections.

Measurable cardinals, when used in this context, have the side effect, albeit non-pathological, of both V and M being HOD.

Corollary 3.1. *The injections h_n are order-preserving, in that given $r_{V_1} <_V r_{V_2}, h_1(r_{V_1}) <_M h_2(r_{V_2})$*

Proof Sketch. This corollary revolves around the claim that $h_n(r_{V_n}) \to r_{M_n}$. As for the orderings $<_V$ and $<_M$, $r_{M_n} <_V r_{M_n}$ is like trying to fit a square peg in a round hole; the orderings of ranks are specific to specific classes of models only. □

Theorem 3.2. *For a tree to be preserved "across injection schemata", it is necessary for said tree to not contain Suslin trees, or be Suslin.*

Proof. If such tree, in which we call T_V, for the representation of V in its T-form, (and its ordering and elements) is preserved across injections, then all of its elements (and subtrees) must be comparable, contradicting the antichain requirement for trees. □

This has many implications for Infinitary Combinatorics, particularly its interaction with General Topology.

Theorem 3.3. *(T_V, τ) in which τ is the topology generated by trees in Definition 2.10, is Hausdorff.*

Proof. Kunen's (2013) definition of Hausdorff-ness for trees is used: (T_V, τ) is Hausdorff iff for all limit κ and $x, y \in \mathcal{L}_\kappa(T)$, if $x \downarrow_V = y \downarrow_V$ then $x = y$.[7] Suppose that $x \neq y$. The proof (via contrapositive) proceeds immediately via a ranking argument. □

Shelah (1977) noted that any tree topology is 1st countable and T_3. The fact that (T_V, τ) is is very obvious, just take Theorem 3.3 and observe regularity. In fact, (T_V, τ) in particular is T_6. And for first countability, take a base for (T_V, τ), and let it observe "open sets", or more aptly, "open collections", in V. T_6-ness for (T_V, τ) implies that (T_V, τ) (and its subsets) are G_δ. Is this preserved via measurable cardinal embeddings in M? We can try using a modified power set operation in M, much like Def in the construction of L. But an issue is encountered; are the same (or even types of) sets in V preserved in M? Again represent M as a tree and endow it with the tree topology as given by Shelah. Use schemata of injections. Define the individual injections h such that $h : S \in V_\alpha \to S_M \in M_\alpha, \alpha = \alpha$, and S is bijective with S_M, and that $V \models \phi(a_n) \iff M \models \phi(h(a_n))$, in which a_n defines (restrictedly) classes in both models.

Remark 5. (On T_V's relation with V itself) T_V is an auxillary set that adds its own relations, constants, and functions to V. Which kinds? To start, we have the the addition of the function schema $f_R : X \in V \to T_V$, which is defined recursively in terms of the von Neumann hierarchy, e.g, α in $V_\alpha := \cup_{\beta<\alpha} P(V_\beta)$. The same goes for other models, with the requirement that they also be transitive. We work in T_V for topological properties instead of V directly primarily for conveinence, but an implication of Hausdorff-ness of trees of V is of a way to effectively "separate" sets in V, in that given 2 sets that are of different rank in V (constructed via different iterations of the power set operation) can be "Hausdorff-ly" be separated.

[7] The notation $x \downarrow$ means the set of elements "below" x in the tree T, and in this case, "below" or "lower" in the ranking.

In retrospect, I realize that this method is essentially a injection-specific method (or variant) of Kunen's iteration of elementary embeddings in his 1971 paper "Elementary Embeddings and Infinitary Combinatorics". A missing requirement for this paper is that $\alpha \in \mathbf{ORD}$. Kunen uses these embeddings to study supercompact cardinals and to prove his inconsistency theorem. In one theorem, he uses his special notion of embedding to weaken the definition of supercompact cardinals. Here are some similarities with Kunen's iterated elementary embeddings and my injection schemata:

(1) Individual quantification over classes, as expressed by the fact that if there is a series of classes in the model V (we work with an embedding $V \to M$), then there is an elementary embedding that embeds from V to M, generated or not, that quantifies over classes individually, but in a schematic manner,
(2) Definition of "sub-functions" of embeddings,
(3) Usage of ordinals and ranks of ordinals ($\alpha \in \mathbf{ORD}$) to construct classes and functions in their respective models. This is essentially a "broadening" of using ranks and levels from V and L to construct classes in said models, but arbitrarily defined for other transitive models.

This injection schema is a special case of the *Reflection Principle*, which states that it is possible to find sets that, with respect to a certain property, "resemble" the class of all sets. Of course, we can treat the injections as sets (ordered pairs), and the property is preservation of rankings/hierarchy in the models. More precisely, in ZFC, we have a schema of formulae $\phi_0, \phi_1, ..., \phi_{n-1}$ in the language $L = \{\in\}$. Let B be a non-empty class and $A(\xi)$ is a set for an $\xi \in \mathbf{ORD}$. Also (Kunen):

(1) $\xi < \eta \to A(\xi) \subseteq A(\eta)$
(2) $A(\eta) = \cup_{\xi<\eta} A(\xi)$ for limit η
(3) $B = \cup_{\eta \in \mathbf{ORD}} A(\xi)$,

so that $\forall \xi \; \exists \eta > \xi \; [A(\eta)) \neq \emptyset$ and $\bigwedge_{i<n} (A(\eta] \preceq_{\phi_n} B)$ and η is a limit ordinal].

Under Kunen, have A represent the function $\mathcal{L}$ and $\mathcal{L}_\alpha$, in which $\alpha \in \mathbf{ORD}$ (and can be replaced with ξ or η), and the schema of formulae as function, injection, set, or class schemata.

We now shift our attention to other elementary embeddings generated by other Large Cardinals.

3.2. **Other Large Cardinals.** Slightly above measurable cardinals are Strong cardinals, in which they result in many, although non-embedding properties. We continue to study stronger Large Cardinals and their topological/tree relation.

Definition 3.3. A cardinal κ is γ-strong iff it is the critical point of some elementary embedding $j : V \to M$ for some transitive class M such that $V_\gamma \subset M$.

Remark 6. In order to "split this up" into an injection schema, have κ be the critical point of either the entirety (together) or the maximal injection. But the fact that the injections preserve ranks is weakened, to avoid conflict with "$V_\gamma \subset M$". An injection is required to have a surjective rank component to keep $V_\gamma \subset M$, making injections not feasible for defining Strong cardinals in terms of trees or in general. Even an arbitrary ranking/construction would still need an injection-surjection. Instead, usage of ultrapowers and extenders would have to be used in order to work with them "in situ", but they cannot (in terms of the current cardinal) be used to create some kind of poset. Therefore, Strong cardinals will not be used in many topological counterexamples.

Woodin cardinals are a generalization and strengthening of γ-strong cardinals, albeit the definition of γ-strongness for Woodin cardinals is modifed. The definition of γ-strong cardinals (for Woodins) again relies on an elementary embedding $j : V \to M$, but the critical point is $V_{\kappa+\gamma} \subseteq M$, and $A \cap V_{\kappa+\gamma} = j(A) \cap V_{\kappa+\gamma}$, and κ in this instance is an uncountable ordinal. κ is $< \delta$-A-strong if it is γ-strong for A and for all $\gamma < \delta$. To be more precise, we say that an inaccessible cardinal δ is *Woodin* if for any set $A \subset V_\delta$, there exists a $\kappa < \delta$ that is $< \delta$-strong in A.[8] It is still possible to "split up" the elementary embedding $j : V \to M$ (in terms of Woodin cardinals) into an injection schema. Have κ be the critical point of the maximum injection. There is no conflict with either $V_{\kappa+\gamma} \subseteq M$ or $j(A) \cap V_{\kappa+\gamma}$, because the critical point does not rely on a direct expansion of ranks of V in terms of embeddings. The previous large cardinal (strong cardinals) relied on that, thus giving a ranking for different V_γ or even sets in V (ultimately for trees) requiring an obviously contradictory injection-surjection. Therefore, one can create a tree-theoretic notion of an elementary embedding $j : V \to M$ but under Woodin cardinals. However, the ordering and rank preservation is edited to meet the needs of $V_{\kappa+\gamma} \subseteq M$, and $A \cap V_{\kappa+\gamma} = j(A) \cap V_{\kappa+\gamma}$. Essentially, we still represent V as a κ-tree like in the original measurable cardinal form of tree-notion embeddings, but with rank functions (functions that assign from V or to M an arbitrary rank) being such that $\restriction V_\alpha, \alpha < \kappa + \gamma$. Thing is, κ is much more variable, with only γ being a potential restrictor (say, the rank of trees in V or M is required to be $< \gamma$), thereby allowing us to well order and partially order ranks of V (and their constituent subsets, such as A) and M, assuming preservation by $j : V \to M$, and create a tree-theoretic notion of the arbitrary ranking of sets in, for example, V or A.

Supercompact cardinals can be given primarily in terms of elementary embeddings. They are essentially a minor strengthening of Measurable cardinals, and a major restriction that they make is of closure of arbitrary sequences of length θ. For example, a (sub-)sequence or injection or function, if it may or may not (although the answer is generally may) contain a sequence of length θ, then it is closed within the elementary embedding $j : V \to M$. κ has very little involvement except the definition of $\theta > \kappa$, and also a restriction on specific sequences in which $k(\alpha) \geq \alpha$ to $\alpha < \kappa$.

Extendible cardinals are not a strengthening nor derived from measurable cardinals, and were instead introduced by Reinhardt (1974), (which was then revised by Silver) who was partly motivated by reflection principles. One can see that Extendible cardinals are chiefly about universal properties, but it is obviously not impossible to produce non-model-theoretic theorems from them.

Vopenka's principle is even more special in that it is not even an elementary embedding from two models of set theory in particular. It relies on another Large

[8] Although Kanamori in his "The Higher Infinite" game multiple equivalent formulations for Woodin cardinals, this will be the preferred formulation for a non-filter notion; usage of the *Woodin filter* will likely not be used very often.

Cardinal to "carry it" to make it a Large Cardinal. It is also a second-order principle, and in first-order ZFC, is a schema of elementary embeddings from submodels/substructures of proper classes of structures within the same language. Therefore, for every natural number n in the meta theory of Vopenka's Principle, there is a formula expressing that Vopěnka's Principle holds for all Σ_n-definable[9] (with parameters) classes. An example of a Σ_n formula is the definition of a compact set in a topological space; to be more precise, it is Σ_1. An example of a Π_n formula is the definition of power set. Although this only matters for classes, there are some interesting implications for simply Σ_n and non-Σ_n formulae, particularly in Topology. Is Vopenka's principle the limit of strength for most topological properties, in that they don't get preserved or even "make sense", definable, in Vopenka? *Given a specific class of model of Set Theory, are there entire universes of topological counterexamples, in which they are not true under Vopenka? Given the formulation of Vopenka via weaker cardinals, do they apply to weaker cardinals too?*

We can also use an alternative formulation of Vopenka's principle given by Kanamori:

Definition 3.4. (Alternative formulation of Vopenka's principle) For any proper class $C \subseteq ORD$, there are $\alpha, \beta \in C$ and a nontrivial elementary embedding $j : \langle V_\alpha, \in, P\rangle \to \langle V_\beta, \in, P\rangle$.

3.2.1. *Trees and some common Topological theorems.* The *Universal Extension Property* states that if X is a normal topological space, and A be a closed subset of X, and let f be a continuous function on A to the closed interval [—1,1].Then f has a continuous extension g which carries X into [— 1,1]. We apply this to V_a. Shelah (1977) proved that if T is a Souslin tree, then its topology is normal, therefore is automatically true for the tree characterized by V_α, (T_{V_α}, τ). As an example, if A is a set "in" the real line, and is closed under the tree topology, then given a mapping from A to [-1,1], this results in an embedding (or function schema) from the real line into [-1,1].

Remark 7. Every set "in" V is clopen under the tree topology.

3.3. **Other than Embeddings.** We turn our attention away from embeddings (especially between models), and look towards more "domestic" functions generated/led by Large Cardinals, and also filters, akin to the proof sketch in the introduction.

3.4. **"The Top of the Hierarchy" and Trees; implication for AC.** Luckily, in 1971, Kenneth Kunen proved his *inconsistency theorem*, with a lemma being of the incompatibility of defining and preserving ranks with an extremely strong large cardinal. Kunen's inconsistency theorem states that there can be no nontrivial elementary embedding from the universe to itself. He proved his inconsistency theorem in second-order Morse-Kelley set theory, namely due to statements involving the satisfaction predicate for class models can be expressed, but another

[9] Σ_n refers to the *Levy Hierarchy*, with $\Sigma_0 = \Pi_0 = \Delta_0$, and a formula A being [3]:

- $\Sigma_{i+1} := A$ is equivalent to $\exists x_1 ... \exists x_n B$ in ZFC, where B is Π_i,
- $\Pi_{i+1} := A$ is equivalent to $\forall x_1 ... \forall x_n B$ in ZFC, where B is Σ_i,

and Δ_{i+1} given that A is equivalent to Σ_{i+1} and Π_{i+1}.

reason is of ease of formulating second-order, statements about embeddings about embeddings-type statements. He then uses such embeddings to construct a combinatorical argument involving first supercompact cardinals, but then strengthening up to $j : V \to V$. Instead of the AC, Skolem functions are used to create a mapping from $M \to M$. The crux of his paper relies on this lemma:[8]

Lemma 3.3.1. *(Kunen)*

(1) $\delta^+ \subset M$.
(2) M *is a transitive model of ZFC.*
(3) i *is an elementary embedding from* M *into* V.
(4) $i(\delta) > \delta$.
(5) $i \restriction \delta$ *is the identity.*
(6) ${}^\omega M \subset M$.

such that there are at least $(2^\omega)^+$ different ordinals, δ, such that there exist M, i with the above properties.

To "generate" the common version of the inconsistency theorem, one has to simply let $M = V$ or similar. AC is inconsistent with $i : V \to V$ because the global axiom of choice (which is needed to define the identity $F :\ {}^\omega V \to V$) can instead be given with Skolem functions, and that for a combinatorical property involving elementary embeddings from the set-theoretic universe to itself[10], Kunen's Lemma (3.3.1) applies. He then uses supercompact and stronger cardinals to show that there are cardinals posessing P, and then later emphasizes that P cannot be proven by the existence of a measurable cardinal. He then "breaks down" the combinatorical property involving elementary embeddings and argues with its individual components $\delta_\zeta, M_\zeta, i_\zeta$ given in the above lemma. This disproves AC in that P is stronger than AC.

There are many formulations of his inconsistency theorem. Here are a few:

(1) There is no elementary embedding $j : L \to L$ from the constructible hierarchy to itself.
(2) There is no nontrivial elementary embedding between two ground models of the universe.
(3) For any definable class D, there is no nontrivial embedding $j : D \to V$, "definable" in this context not referring to the definability relation in L, but definable from elements/sets of V.

This is why one must be *very careful* in constructing/borrowing symbols/constructions in M from V when constructing an (ad-hoc) tree or ordering in M from V. *Only* the order is ad hoc and borrowed from V.

This is also why Large Cardinals that are inconsistent with AC will not be discussed in this paper; AC is trivial for many Topological constructions, and using such Large Cardinals is simply "cheating" to get a result.[11]

[10]In Kunen's original paper, this was just a vague combinatorical property involving elementary embeddings, but he most likely did not have this implication in mind.

[11]For further study on equivalents to AC, read "Equivalents of the Axiom of Choice" by Herman and Jean E. Rubin.

4. Counterexamples

4.1. The Real Line and Compactifications. We start off with a simple theorem from General Topology.

Theorem 4.1. *Under a measurable cardinal, the Stone-Cech Compactification of the Real Line is "preserved", in that it is still possible under the embedding $j : V \to M$, with M transitive and κ and the crit. point of j. The Stone-Cech Compactification of a space is βX, of a space X, and is largest, most general compact Hausdorff space generated via maps from X to βX.*

Proof. (With Embeddings and Trees) We use a modified "unit interval" (which is [-1,1], but is still isomorphic to [0,1]) to create the Stone-Cech Compactification. Let X, the topological space, be constructed "von-Neumann-ly" in V to create the natural numbers, and then construct the rational numbers via equivalence classes of natural numbers. From there, we can construct the real numbers via either Cauchy sequences or Dedekind cuts; our choice of construction is irrelevant. We use the Euclidean Topology. Suppose that C is the set of all continuous functions from $X \to [0,1]$, and a function $f \in C$ is such that $f_{r_{V_n}} \restriction V_n$, or $f_{r_{M_n}} \restriction M_n$, or $f_{V_n} \restriction V_n$ (same eventually applies for M_n and for other models; we can have $f_{r_{M_n}} \restriction V_n$ or with V_n and r_{M_n} reversed, so long as the the injection schema from V to M and respective ranks are preserved via $f \in C$). Consider the map $e : X \to [0,1]^C$, and in which $e(x) : f \to f(x)$ (standard definition for a unit interval construction). The Tietze Extension Theorem applies doubly for both the Real Line (as it is normal too) and V and M (and their respective trees), in fact for the measurable cardinal κ. What are some implications for this? One can doubly nest both a "shrinking" or special embedding/function from a specific level of subclass of T_V into the interval $[-1,1]$, and the real line can be shrunken into the same $[-1,1]$. We now modify the "vanilla" construction and replace the unit interval with the interval $[-1,1]$, and C (the set of all continuous functions) and modify its definition to "the set of all continuous functions such that they obey order of trees of models". Represent the modified C by Co, with o standing for "obedience". We also modify the Product Topology for this proof; as with the usage of ordinals and ranks of ordinals to construct classes and functions in injection schemata, we add the fact that $i \in I$ of the index set I is required to be an element of either T_V or T_M, and the ordinality of i is $< \kappa$. Endow $[-1,1]^{Co}$ with the modified Product Topology of the topological subspaces generated via functions caused by the function schema from $V \to M$. As a remark, these additonal endow-ment of topologies and sets of continuous functions are a result of the addition of relation, function, and constant symbols to both models. Construct an elementary equivalence between the "vanilla" Stone-Cech Compactification and the "modified", measurable, Stone-Cech Compactification. Hausdorff and compact-ness of $\beta\mathbb{R}$ in V and M, and even in the "vanilla" model, without the embedding, is not an issue, particularly regarding the additional tree construction provided; all trees endowed with the tree topology (which is further extended to measurables) are Hausdorff and are compact anyway. To check that this is an extension or preservation of an already existing compactification and is not due to forcing or some other property, trees are already inherent in our measurable cardinal construction, and therefore such theorems. The original compactification of $\mathbb{R}$ without measurables is letting C (un-modified) to equal all open sets from X into $[0,1]$ (which is continuous); this is somewhat of a widening (yet still valid) construction of embeddings. As the closure of X in $[0,1]^C$ is βX (replaced with $\mathbb{R}$), so we have $[-1,1]^{Co}$ and $\mathbb{R}$'s closure in it also being $\beta\mathbb{R}$.

(With Ultrafilters) We proceed in the footsteps of the proof of that homogeneity of pairs of filters is guaranteed, but major adjustments are made. (Ultra)filters can effectively generalize topological properties, particularly on *cluster points*.

Definition 4.1. (Cluster points and filters) A point s of a filter F of a topological space X is a cluster point of the filter F iff F is frequent in every neighborhood of F, and F is an arbitrary filter. Note that the filter F is the neighborhood basis of x.

This is generalized to the Stone-Cech Compactification in that cluster points can elegantly generate compactifications; just set the open neighborhoods of the cluster points in the new topological space βX to be the space's basis.

We use the following theorem: A topological space is a Hausdorff space if and only if each net in the space converges to at most one point (Kelley), and then adopt it for filters, given that nets' and filters' theorems are identical.

Set the topological space to be $(\mathbb{R}, \tau)$ in which τ is the Euclidean Topology. Also, suppose that the filters F of the topological space:

$$(\mathbb{R}, \tau) \text{ be such that } F \subset \kappa\text{-complete nonprincipal ultrafilter(s) on } \kappa.$$

Select a point s of F which is a cluster point of F. We already know that the resulting compactification is Hausdorff because we can construct nets and filters containing cluster points of F such that they converge to at most one point. The fact that the resulting compactification is the largest (and general) is from the filter being the limit of κ.

□

Remark 8. Measurable cardinals are the least annoying Large Cardinal to work with in General Topology, particularly in this paper. One very elegant property is that it results in $P(\kappa)^V = P(\kappa)^M$, furthering the ease of preservation of ranks and hierarchies. [2]

Corollary 4.2. *The one-point compactification of the Real Line is "preserved" under a measurable cardinal (usage of word "preserved" as in Theorem 4.1).*

And as a strengthening,

Corollary 4.3. *For any compactification of the Real Line, it is preserved under a measurable cardinal.*

The proof is trivial and comes from the preservation of homomorphisms of (particularly sets in the Real Line) via measurables.

We now climb up the "ladder" (or web, but that is not a practical analogy) of Large Cardinals. How high can we go until it breaks?

Theorem 4.4. *Under a strong cardinal, the construction of the Real Numbers is preserved and goes "as usual".*

Proof. Proceed via extenders, in lieu of Remark 5.

Definition 4.2. A cardinal κ is *strong* iff it is uncountable and for every set X of rank $\lambda < \kappa$, there is a $(\kappa, \beth_\lambda^+)$-extender E such that, letting the ultrapower[12] of V by E be called Ult_E and the canonical elementary embedding from $V \to Ult_E$ be j, $X \in Ult_E$ and $\lambda < j(\kappa)$. (Initially, a notion of a recursive construction of ultrapowers and extenders was going to be used, but it was realized that the notion of a "cardinal/set/hierarchy rank" is already contained in the definition.)

The base set X for the ultrapower construction is a set with the ability to satisfy the Peano Axioms and ZFC. In the ultrapower construction, let the $(\kappa, \beth_\lambda^+)$-extender be defined as an elementary embedding of ZFC. Taking E_α a constituent ultrafilter of $(\kappa, \beth_\lambda^+)$, we use the notion of $=_{E_\alpha}$-equivalence, and additionally E_α-equivalences of functions/sets, denoted $[f]$. As we already have a base set X which satisfies Peano, the models M_i generated and contained in the ultrapower also satisfy Peano, which means we can construct the natural numbers in them. Under the extender, the canonical elementary embedding from $V \to Ult_E$ is preserved, therefore the preservation of the construction of the natural numbers from strong cardinals. The construction of the natural numbers is also preserved under Ult_E.

As for rationals, construct an equivalence relation between any 2 natural numbers. Then, we can use E_α-equivalences for equivalence relations between any 2 natural numbers, i.e. an E_α-specific version of such relation. Finally, to construct Cauchy sequences of the rationals, simply use filters $F \subset E_\alpha$ and nets constructed from F to create sequences of the rationals. Then assign each sequence (a_n) or (b_n) a rank (which represents a real number) in E_α. The construction of the Reals from (Cauchy) sequences goes as usual, but with the observation of the ultrapower Ult_E and the canonical elementary embedding from V to Ult_E. □

Theorem 4.5. *It is possible to endow $\mathbb{R}$ with the Euclidean Topology while working in ZFC + Strong.*

Proof Sketch. Like a run-of-the-mill analysis proof, but with functions/sequences observing/elements of the E_α-equivalences of functions/sets of the $(\kappa, \beth_\lambda^+)$-extender, or it being preserved (holds in both) V and Ult_E via the canonical elementary embedding $j : V \to Ult_E$. □

Theorem 4.6. *The Stone-Cech Compactification of the Real Line is preserved using the Strong cardinal.*

Proof.

Remark 9. This does not work for ultrafilters and the method below solely.[13]

(Ultrafilters) Proceed via the ultrapower construction of the Stone-Cech Compactification. It must be shown that the ultrafilters used to construct the ultrapower

[12]An ultraproduct is a quotient of the direct product of a family of structures. The *ultrapower* is the special case in which all the structures are equal. The quotient is $=_{E_\alpha}$. Here, the structures refer to models of set theory or filters; reference to models or filters depends on the context.

[13]Also, the original form of this theorem is "The Stone-Cech Compactification of the Real Line is not preserved using the Strong cardinal", before realizing the usage of maps and universal properties in this proof.

Ult_E of the ultrapower of V by E (the $(\kappa, \beth_\lambda^+)$-extender E) cannot have the Stone topology constructed on it. The *Stone topology on* $\mathbb{R}$ is generated by sets of the form $\{U_\alpha : U_\alpha \in F\}$ for U a subset of $\mathbb{R}$. But this ultrafilter construction depends on the discreteness of $\mathbb{R}$. But $\mathbb{R}$ is not discrete.

(Universal Properties and Functoriality)

Definition 4.3. Let a continuous map $i_\mathbb{R} : \mathbb{R} \to \beta\mathbb{R}$ have the universal property such that for any continuous map $f : \mathbb{R} \to K$, in which K is a compact Hausdorff space, that it extends uniquely to a continuous map $\beta f : \beta\mathbb{R} \to K$. Define this map $i_\mathbb{R}$ as another form of the Stone-Cech Compactification of $\mathbb{R}$.

(Ad Hoc Definition) Let the map $i_\mathbb{R}$ be an element of the canonical elementary embedding from $V \to Ult_E$, and $\mathbb{R} \subset V$ and $\beta\mathbb{R} \subset Ult_E$. Obviously $R \subset V$, and $\beta\mathbb{R} \subset Ult_E$ because of

Lemma 4.6.1. *(Lemma of Definition 3.1 and Remark 4, also lemma of arbitrarily defining trees from models) Submodels of Ult_E can have a ad hoc topology constructed on them, given that open sets of submodels are given by an ad hoc order/ranking in said submodels according to Definition 3.1.*

Also, suppose a continuous map f does not extend to a continuous map βf : $\beta\mathbb{R} \to K$. Have this "extension" be a "second-order mapping", given in terms of the extender E. But given that the models are arbitrarily ranked, then trees can be constructed out of them. Also, all arbitrarily ordered trees (endowed with the Tree Topology) are T_3 and compact. "There are no continuous maps between 2 Hausdorff trees" is additionally false. □

Corollary 4.7. *The Stone-Cech compactification for Topological spaces in general is preserved using Strong cardinals.*

Let us go higher.

Theorem 4.8. *The Stone-Cech compactification for Topological spaces is preserved under the Woodin cardinal.*

Proof. See the paragraph on page 8 about Woodin cardinals; if we choose the filter formulation of Woodin cardinals, we use Lemma 4.6.1, else use the fact of construction of trees via Woodin cardinals and their subsequent compactification, similar to the proof of Theorem 4.6. □

Theorem 4.9. *The Stone-Cech compactification for Topological spaces is preserved under a supercompact cardinal.*

Theorem 4.10. *The Stone-Cech compactification for Topological spaces is preserved under Huge cardinals.*

Remark 10. We primarily use the elementary embedding notion to prove this theorem. Choices of V and M are irrelevant. Also, by "huge", we are referring to n-huge cardinals, almost n-huge, n-huge with target λ, and almost n-huge with target λ.

Theorem 4.11. *The Stone-Cech compactification for Topological spaces is preserved under (super) n-huge cardinals.*

Theorem 4.12. *The Stone-Cech compactification for Topological spaces is preserved under ultrahuge and hyperhuge cardinals.*

Theorem 4.13. *The Stone-Cech compactification for uncountable Topological spaces is not preserved under any rank-into-rank axiom.*

Remark 11. Even if Kunen's Inconsistency Theorem is false, this still contradicts the "V-centric" nature of the Stone-Cech compactification; it cannot be done in L, even with L having a "ranking" as well. This is with assuming $V \neq L$ or even $V = L$[14] Even with $V \neq L$, $V_\omega = L_\omega$, and generally for uncountable κ, $L_\kappa \subseteq V_\kappa$; some sets of L_κ are missing. Therefore, there are always some "gaps" in uncountable Topological spaces in L.

Theorem 4.14. *Theorems 4.6 to 4.13 also apply to any compactification.*

Theorem 4.15. *A compactification for any Topological space is preserved under any Large Cardinal weaker than Rank-into-Rank.*

4.2. **The Strong Ultrafilter Topology.** This is counterexample (113) in "Counterexamples in General Topology". The Integers are not a concern, given their easy construction in V. Take (113-1) as a good example: any Large Cardinal that systematically uses ultrafilters on models is required to interact with a specific net/filter on topological spaces.

Actually, we can weaken this to a measurable cardinal.

Theorem 4.16. *Under (113-1) of "Counterexamples in General Topology", Z_N is Hausdorff under a Measurable cardinal. Let Z^+ be the positive integers, and let N_U*[15] *be the collection of all ultrafilters on Z^+. Let $Z_N = Z^+ \cup N_U$, and let the topology r on Z_N be generated by the points of Z^+ together with all the sets of the form*[16] *$A \cup F$ where $A \in F \in M$.*

Proof. (Using Elementary Embeddings) Construct the set of positive integers in V the usual way. An example of a filter on Z^+ is:

$$\mathcal{B}_\prime = \ ...\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,5\},...$$

and $\mathcal{B}_\backslash$ be filters constructed iteratively and similarly; we can define N_U (an ultrafilter) as:

$$N_U = \mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, ...$$

in which individual $\mathcal{B}_n$ vary, in that they can be either $\{5,7,9\},...$ or differ, but must be an arithmetic/geometric sequence. It must be proven that for two constituent $\mathcal{B}_n$ in N_U, that they are incomparable. This proves that Z_N is Hausdorff under the elementary embedding $j : V \to M$.

M can still construct the natural numbers. Elements of and ultrafilters themselves get preserved under the elementary embedding because:

(1) it preserves ranks and rankings of elements, which is nescessary for ordering/ranking the ultrafilter(s);

[14]This just says that any set in V is also constructible by L's Def relation, not so much that $V_\kappa = L_\kappa$.

[15]In the original book, the collection of all nonprincipal ultrafilters on Z^+ is represented via M, but obviously both M and N are overused. Also, non-principality of ultrafilters is not nescessary.

[16]F in this context is not a filter, but such possibility can be explored.

(2) and even given schemata of injections (representing the elementary embedding) from $N_U|V \to N_U|M$ (assuming between individual ultrafilters), $N_U|M$ is still an ultrafilter.

We can then conclude that given a family of ultrafilters of N_U, two ultrafilters of it are incomparable (in the ranks of the models). Given $A \in \mathcal{B}_{\mathcal{A}} - \mathcal{B}_{\mathcal{B}}$ and $B \in \mathcal{B}_{\mathcal{B}} - \mathcal{B}_{\mathcal{A}}$, A' and B' are separated because $B' \in \mathcal{B}_{\mathcal{A}}$ and $A' \in \mathcal{B}_{\mathcal{B}}$. Generalize to A and B in that:

$$A \in (\mathcal{B}_{\mathcal{A}} - \mathcal{B}_{\mathcal{B}}) \cap \mathcal{B}_{\mathcal{A}}$$

and

$$B \in (\mathcal{B}_{\mathcal{B}} - \mathcal{B}_{\mathcal{A}}) \cap \mathcal{B}_{\mathcal{B}}.$$

Then, show that $(A - B) \cup \{\mathcal{B}_0\}$ and $(B - A) \cup \{\mathcal{B}_1\}$ are disjoint neighborhoods of $\mathcal{B}_0$ and $\mathcal{B}_1$.

Separation of sets has been proven so far. This argument can very be easily generalized to individual points of the form $A \cup F$, and also preserved under the Measurable cardinal's embedding.

(Using Ultrafilters) Construct a non-principal ultrafilter on the critical point of the elementary embedding $j : V \to M$; the rest goes as the proof of Theorem 4.17. □

Remark 12. (113-1) in Theorem 4.16 is actually a T_3 space. It is not a normal space because $\mathcal{B}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{B}}$ are not obligated to be open.

Theorem 4.17. *Under (113-1) of "Counterexamples in General Topology", Z_N is Hausdorff under a Supercompact cardinal.*

Proof. (Using Ultrafilters) Construct a non-principal ultrafilter on the critical point of the elementary embedding $j : V \to M$. Define the canonical principal ultrafilter P_U by:

$$A \in P_U \iff A \subseteq \kappa \text{ and } \kappa \in j(A).$$

U is a κ-complete non-principal ultrafilter over κ. U has the additional property that it is comprised of arbitrary sequences of N^+ of length θ. Proceed via induction on the arbitrary sequences of length θ, in which the "starting point" is $\theta = \omega_1$. In particular, show that if there exist $\mathcal{B}_0, \mathcal{B}_1$ of N_U (the collection of non-principal ultrafilters on Z^+) which are comprised of arbitrary sequences of length θ, that if there exist $(A - B) \cup \{\mathcal{B}_0\}$ and $(B - A) \cup \{\mathcal{B}_1\}$, they are disjoint neighborhoods of $\mathcal{B}_0$ and $\mathcal{B}_1$.[17] □

Theorem 4.18. *Under (113-1) of "Counterexamples in General Topology", Z_N is Hausdorff under an Extendible cardinal.*

Proof. We use ultrafilters. Construct an ultrafilter P_U on the critical point of the elementary embedding $j : V_{\kappa+\eta} \to V_\theta$ in the same manner as Theorem 4.17. This is how the ultrafilter P_U is precisely constructed on the above elementary embedding:

[17] Most details are omitted from Stenn and Seeback's orignal proof.

$$A \in P_U \iff A \subseteq \kappa + \eta \text{ and } \kappa \restriction \theta \in j(A).$$

and also,

$$\theta \restriction \kappa + \eta$$

in which θ is an ordinal, and this equation applies for sets/filters of Z^+.

Work via induction on θ. If $\theta = 0$, then this obviously fails. If $\theta = \eta + 1$ or $\theta) = \lambda + 1$, then given $\eta + 1 \restriction \kappa + \eta$ for elements/sequences of the ultrafilter (with η representing elements/sequences in terms of cardinality), they are below κ, and we are essentially left with $|A| = n + 1/\kappa + \eta$, with n being the "original" cardinality of the constituent sets. As an example of an ultrafilter constructed in this manner, we could have:

$$\mathcal{B} = \; ...\{1, 2\}, \{2, 3\}, \{3, 4\}...$$

given $\kappa = 1$, $\eta = 1$, and $n = 1$.

But this applies for κ, η, and n only if they result in a natural number cardinality. In that case, such ultrafilters are incomparable. In the other case, the assumption that such ultrafilters are comparable is vacauously true. To make a collection of ultrafilters "work" in N^+, obviously have n divisible by $\kappa + \eta$.

□

Remark 13. Note that dividing n by $\kappa + \eta$ might not be allowed in some constructions; in particular Vopenka's principle/cardinal.

Theorem 4.19. *In (113-1) of "Counterexamples in General Topology", Z_N is not Hausdorff under Vopenka's principle.*

Proof. Definition 3.4 is used. Suppose that the original construction of the Strong Ultrafilter Topology of N^+ is done in $\langle V_\alpha, \in, P\rangle$. Does $\langle V_\alpha, \in, P\rangle \to \langle V_\beta, \in, P\rangle$ still preserve (1) and (2) in the proof of Theorem 4.16? For (1), note that an element a of $\langle V_\alpha, \in, P\rangle$ with rank r will get "pushed" to another rank $r+$, in which $r < r+$ in $\langle V_\beta, \in, P\rangle$. A question that appears is, "does this new ranking still preserve the order and ranking of the elements of the ultrafilters?", in which the answer is "not nescessarily".

Of course, all of this is done via schemata of injections, representing the elementary embedding, to avoid having to use second-order logic.

$$a \in \langle V_\alpha, \in, P\rangle \xrightarrow{i_j \in j} b \in \langle V_\beta, \in, P\rangle$$

$$b \in \langle V_\alpha, \in, P\rangle \xrightarrow{i_j \in j} a \in \langle V_\beta, \in, P\rangle$$

$$c \in \langle V_\alpha, \in, P\rangle \xrightarrow{i_j \in j} c \in \langle V_\beta, \in, P\rangle$$

(Figure 1. Illustration of Proof of Theorem 4.19, on the impossibility of the preservation of rankings under schemata of injections. The elements of $\langle V_\alpha, \in, P\rangle$ are ordered top-down, largest-to-smallest, by their rank. $i_j \in j$ represents an individual injection.)

and trivially, (2) is not true in the proof of Theorem 4.16.

(Using Vopenka filters)

Definition 4.4. (Kanamori) A *Vopenka filter* is a subset of κ such that $\kappa - X$ is not Vopenka in κ.

Take X to be N^+. But the Vopenka filter (as given by Kanamori) is not ultra, because it is "restricted" and not containing κ.

□

Remark 14. Hausdorff-ness is Σ_0 on the Levy Hierarchy, thus Vopenka also applies to it. Even if a purely unbounded formulation of Hausdorff-ness were to be given, it would still be Σ_2.

4.3. **Homogeneity of Spaces.**

Theorem 1 is that for b comprised of pairs of an ultrafilter $\mathcal{B}$, it is homogeneous under a measurable cardinal.

Theorem 1. A filter F is not homogeneous given that such filter is part of an ultrapower generated by a measurable cardinal.

Theorem 4.20. *If X is a zero-dimensional space that is counted as first, then X^ω is homogeneous for a measurable cardinal.*

Proof. Set F as a filter of a κ-complete nonprincipal ultrafilter on κ, and F also as the filter of the first-countable zero-dimensional space. As we know that for a net on a first-countable zero-dimensional space is homogeneous (given that it contains a sequence of neighborhoods $N_1, .., N_n$ such that for any open neighborhood N of x there exist an integer i with N_i contained in N, and also that for any pair of neighborhoods (N_1, N_2), the net maps it (individual points in the neighborhoods) to another pair, individually for each point), we can easily extend this to a filter. And trivially, this filter can be a κ-complete nonprincipal ultrafilter on κ, or generated via the critical point of the elementary embedding $j : V \to M$. □

Theorem 4.21. *If X is a zero-dimensional space that is first-countable, then X^ω is not homogeneous under an Extendible cardinal.*

Proof Sketch. Proceed like in the proof of Theorem 4.18, but observe that the division of n by $\kappa + \eta$ is antithetical to a filter such that it is the filter that can be readily extended to a net like in the proof of Theorem 4.20. □

Theorem 4.22. *If X is a zero-dimensional space that is first-countable, then X^ω is not homogeneous for Vopenka's principle.*

Theorem 4.23. *If X is a zero-dimensional space that is first-countable, then X^ω is homogeneous under a Supercompact cardinal.*

4.4. **The "Either-Or" Topology.**

Definition 4.5. The *either-or topology* is a a topology defined on the closed interval X = $[-1, 1]$ by declaring a set open if it either does not contain $\{0\}$ or does contain $(-1, 1)$.

The either-or topology was generated by Stenn and Seebach as an ad-hoc example for Topological counterexamples. Here are some properties of X:

(1) X is compact.
(2) X is Lindelof.
(3) X is first countable.
(4) X is Hausdorff, and is actually T_5.
(5) X is non-regular.

Theorem 4.24. *(1)-(5) are all preserved under a Measurable cardinal.*

Proof Sketch. (1) If compactness of X were not to be preserved under the embedding $j : V \to M$, then M or j must "delete" some finite subcovers of some collection of open covers for a topological space constructed in V. But then j must delete some sets in M. But then this makes j surjective. (2) Weakening of (1). (3) There exists an ultrafilter U generated from the measurable cardinal with neighborhoods in U such that first countability applies. (4) and (5) are trivial, and come from trees of measurable cardinals. □

Theorem 4.25. *(1), (2), and (4) are all not preserved under Vopenka's Principle.*

5. Conclusion

This material is aptly named *Part I* because this paper has left out many topological properties and counterexamples. Only 4 counterexamples and properties were given; yet 9 pages were taken to prove/show them, and the underlying machinery to make these proofs take up another 10 pages. Other counterexamples, including those not given in "Counterexamples in General Topology", will be given in another paper.

So far, we have proven and shown the following:

- Measurable cardinals fulfill and preserve many, and possibly a large number of Topological properties and counterexamples.
- Strong cardinals also preserve many Topological properties, but very likely less so than Measurables.
- Large cardinals under AC are the limit of strength of preservation of Topological properties, but Vopenka's Principle (and its resulting cardinal), and even Extendible cardinals, can sometimes be an upper limit for the strength and preservation of Topological properties.
- Potential implications of the Levy Hierarchy of Topological properties and their interaction with Vopenka's Principle: do/can they generate entire "bubbles" or "universes" of Topological counterexamples? In fact, the Levy Hierarchy and their categorization of Topological properties on it and their properties can serve as very good ammunition for another paper, particularly their interaction with other Large Cardinals.
- Notion of giving Elementary Embeddings in terms of Injection schemata, but this is not entirely original; Kunen did the same thing, but it was originally in terms of "iterated elementary embeddings".
- The tree-theoretic notion of V and other models and usage of this for Large Cardinal embeddings.

A wealth of open questions have been inspired by this paper, and it is encouraged for the reader to reflect on them, to stimulate the study of Topological counterexamples via Large Cardinals:

Question 5.1. What are some examples of Topological properties in which there is or is not a formula expressing that Vopenka's Principle holds for them?

Question 5.2. Can we apply some of the methods used in this paper to, say, Algebra or Algebraic Topology?

Question 5.3. If Kunen's Inconsistency Theorem were false, how much would the limit on Large Cardinal strength implying/observing/preserving extend beyond the least Large Cardinal inconsistent with AC?

We already see how for Theorem 4.13, the non-application of Kunen Inconsistency does not affect it much.

Question 5.4. Are there any Topological properties in which a weaker Large Cardinal axiom (weaker in this case being weaker than Measurable) is not preserved?

I would also like to thank anonymous internet users for helping to contribute, in part, the content of this paper.

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